SOLUTIONS OF TRIANGLE

The process of calculating the sides and angles of triangle using given information is called solution of triangle. In a ΔABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.

1. SINE FORMULAE :

In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and Δ is area of triangle.

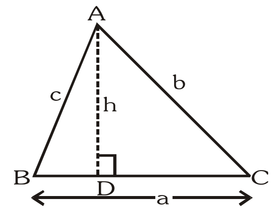


Illustration 1 : Angles of a triangle are in 4 : 1 : 1 ratio. The ratio between its greatest side and perimeter is

(A) $\frac{3}{2+\sqrt{3}}$ (B) $\frac{\sqrt{3}}{2+\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2-\sqrt{3}}$ (D) $\frac{1}{2+\sqrt{3}}$

Solution : Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120 , 30 , 30 .

If sides opposite to these angles are a, b, c respectively, then a will be the greatest side. Now from sine formula $\frac{a}{\sin 120°} = \frac{b}{\sin 30°} = \frac{c}{\sin 30°}$

$\Rightarrow \frac{a}{\sqrt{3}/2} = \frac{b}{1/2} = \frac{c}{1/2}$

$\Rightarrow \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1} = k$ (say)

then a = $\sqrt{3}k$, perimeter = $(2+\sqrt{3})k$

$\therefore$ required ratio = $\frac{\sqrt{3}k}{(2+\sqrt{3})k} = \frac{\sqrt{3}}{2+\sqrt{3}}$ **Ans. (B)**

Illustration 2 : In triangle ABC, if b = 3, c = 4 and $\angle B = \pi/3$, then number of such triangles is -

(A) 1 (B) 2 (C) 0 (D) infinite

Solution : Using sine formulae $\frac{\sin B}{b} = \frac{\sin C}{c}$

$\Rightarrow \frac{\sin \pi/3}{3} = \frac{\sin C}{4} \Rightarrow \frac{\sqrt{3}}{6} = \frac{\sin C}{4} \Rightarrow \sin C = \frac{2}{\sqrt{3}} > 1$ which is not possible.

Hence there exist no triangle with given elements. **Ans. (C)**

Illustration 3 : The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Solution : Let the sides be n, n + 1, n + 2 cms.

i.e. AC = n, AB = n + 1, BC = n + 2

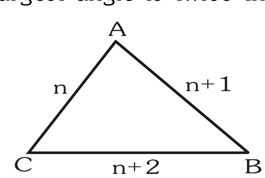
Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180$

$\Rightarrow 3\angle B + \angle C = 180 \Rightarrow \angle C = 180 - 3\angle B$

We have, sine law as,



$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \Rightarrow \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180-3B)}{n+1}$$

$\Rightarrow \quad \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin 3B}{n+1}$
(i) (ii) (iii)

from (i) and (ii);

$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n} \quad \Rightarrow \quad \cos B = \frac{n+2}{2n}$ .... (iv)

and from (ii) and (iii);

$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1} \quad \Rightarrow \quad \frac{\sin B}{n} = \frac{\sin B(3-4\sin^2 B)}{n+1}$

$\Rightarrow \quad \frac{n+1}{n} = 3 - 4(1-\cos^2 B)$ .... (v)

from (iv) and (v), we get

$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2 \quad \Rightarrow \quad \frac{n+1}{n} + 1 = \left(\frac{n^2+4n+4}{n^2}\right)$

$\Rightarrow \quad \frac{2n+1}{n} = \frac{n^2+4n+4}{n^2} \quad \Rightarrow \quad 2n^2 + n = n^2 + 4n + 4$

$\Rightarrow \quad n^2 - 3n - 4 = 0 \quad \Rightarrow \quad (n-4)(n+1) = 0$

$n = 4$ or -1

where $n \neq -1$

$\therefore \quad n = 4$. Hence the sides are 4, 5, 6 **Ans.**

Do yourself - 1 :

(i) If in a ΔABC, $\angle A = \frac{\pi}{6}$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.

(ii) Show that, in any ΔABC : $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$.

(iii) If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, show that a^2, b^2, c^2 are in A.P.

(iv) If in a ΔABC, $\angle A = 3\angle B$, then prove that $\sin B = \frac{1}{2}\sqrt{\frac{3b-a}{b}}$.

2. COSINE FORMULAE :

(a) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ **(b)** $\cos B = \frac{c^2+a^2-b^2}{2ca}$ **(c)** $\cos C = \frac{a^2+b^2-c^2}{2ab}$

or $a^2 = b^2 + c^2 - 2bc\cos A$

Illustration 4 : In a triangle ABC, if B = 30 and c = $\sqrt{3}$ b, then A can be equal to -

(A) 45 (B) 60 (C) 90 (D) 120

Solution : We have $\cos B = \frac{c^2+a^2-b^2}{2ca} \Rightarrow \frac{\sqrt{3}}{2} = \frac{3b^2+a^2-b^2}{2\times\sqrt{3}b\times a}$

$\Rightarrow \quad a^2 - 3ab + 2b^2 = 0 \quad \Rightarrow \quad (a-2b)(a-b) = 0$

$\Rightarrow$ Either $a = b \quad \Rightarrow \quad A = 30$

or $a = 2b \quad \Rightarrow \quad a^2 = 4b^2 = b^2 + c^2 \quad \Rightarrow \quad A = 90$. **Ans. (C)**

Illustration 5 : In a triangle ABC, $(a^2 - b^2 - c^2)\tan A + (a^2 - b^2 + c^2)\tan B$ is equal to -

(A) $(a^2 + b^2 - c^2)\tan C$ (B) $(a^2 + b^2 + c^2)\tan C$

(C) $(b^2 + c^2 - a^2)\tan C$ (D) none of these

Solution : Using cosine law :

The given expression is equal to $-2\,bc\cos A\tan A + 2\,ac\cos B\tan B$

$$= 2abc\left(-\frac{\sin A}{a}+\frac{\sin B}{b}\right)=0$$

Ans. (D)

Illustration 6 : If in a triangle ABC, $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ac}$, find the $\angle A$ =

(A) 90 (B) 60 (C) 30 (D) none of these

Solution : We have $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ac}$

Multiplying both sides of abc, we get

$\Rightarrow$ $2bc\cos A + ac\cos B + 2ab\cos C = a^2 + b^2$

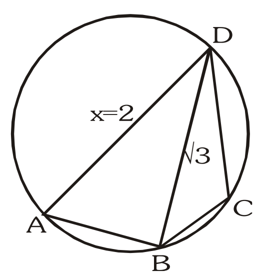
$\Rightarrow$ $(b^2 + c^2 - a^2) + \frac{(a^2+c^2-b^2)}{2}+(a^2+b^2-c^2)=a^2+b^2$

$\Rightarrow$ $c^2 + a^2 - b^2 = 2a^2 - 2b^2$ $\Rightarrow$ $b^2 + c^2 = a^2$

$\therefore$ ΔABC is right angled at A. $\Rightarrow$ $\angle A = 90$ **Ans. (A)**

Illustration 7 : A cyclic quadrilateral ABCD of area $\frac{3\sqrt{3}}{4}$ is inscribed in unit circle. If one of its side AB = 1, and the diagonal $BD=\sqrt{3}$, find lengths of the other sides.

Solution : AB = 1, $BD=\sqrt{3}$, OA = OB = OD = 1



The given circle of radius 1 is also circumcircle of Δ ABD

$\Rightarrow$ R = 1 for ΔABD

$\Rightarrow$ $\frac{a}{\sin A}=2R$ $\Rightarrow$ A = 60

and hence C = 120

Also by cosine rule on ΔABD, $\left(\sqrt{3}\right)^2=1^2+x^2-2x\cos 60°$

$\Rightarrow$ x = 2

Now, area ABCD = ΔABD + ΔBCD

$\Rightarrow$ $\frac{3\sqrt{3}}{4}=\frac{1}{2}(1.2.\sin 60°)+\frac{1}{2}\left(c.d.\sin 120°\right)$

$\Rightarrow$ cd = 1, or $c^2d^2 = 1$

Also by cosine rule on triangle BCD we have

$\left(\sqrt{3}\right)^2=c^2+d^2-2cd\cos 120°=c^2+d^2+cd$

$\Rightarrow$ $c^2 + d^2 = 2$ or cd = 1

$\Rightarrow$ c^2 and d^2 are the roots of $t^2 - 2t + 1 = 0$

$\therefore$ $c^2 = d^2 = 1$ $\therefore$ BC = 1 = CD and AD = x = 2.

Do yourself - 2 :

(i) If a : b : c = 4 : 5 : 6, then show that $\angle C = 2\angle A$.

(ii) In any ΔABC, prove that

(a) $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

(b) $\frac{b^2}{a}\cos A+\frac{c^2}{b}\cos B+\frac{a^2}{c}\cos C=\frac{a^4+b^4+c^4}{2abc}$

3. PROJECTION FORMULAE :

(a) b cos C + c cos B = a **(b)** c cos A + a cos C = b **(c)** a cos B + b cos A = c

Illustration 8 : In a $\Delta ABC, c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}=\frac{3b}{2}$, then show a, b, c are in A.P.

Solution : Here, $\frac{c}{2}(1+\cos A)+\frac{a}{2}(1+\cos C)=\frac{3b}{2}$

$\Rightarrow$ a + c + (c cos A + a cos C) = 3b

$\Rightarrow$ a + c + b = 3b {using projection formula}

$\Rightarrow$ a + c = 2b

which shows a, b, c are in A.P.

Do yourself - 3 :

(i) In a ΔABC, if $\angle A=\frac{\pi}{4}$, $\angle B=\frac{5\pi}{12}$, show that $a+c\sqrt{2}=2b$.

(ii) In a ΔABC, prove that : (a) $b(a\cos C - c\cos A) = a^2 - c^2$ (b) $2\left(b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}\right)=a+b+c$

4. NAPIER'S ANALOGY (TANGENT RULE) :

(a) $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$ **(b)** $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$ **(c)** $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

Illustration 9 : In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Solution : Here, $\tan\left(\frac{A-B}{2}\right)=\frac{1}{3}\tan\left(\frac{A+B}{2}\right)$ .... (i)

using Napier's analogy, $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$ .... (ii)

from (i) & (ii) ;

$\frac{1}{3}\tan\left(\frac{A+B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right) \Rightarrow \frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$

{as A + B + C = π $\therefore$ $\tan\left(\frac{B+C}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}$ }

$\Rightarrow$ $\frac{a-b}{a+b}=\frac{1}{3}$ or 3a – 3b = a + b

$2a = 4b$ or $\frac{a}{b} = \frac{2}{1} \Rightarrow \frac{b}{a} = \frac{1}{2}$

Thus the ratio of the sides opposite to the angles is b : a = 1 : 2. **Ans.**

Do yourself - 4 :

(i) In any ΔABC, prove that $\frac{b-c}{b+c} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$

(ii) If ΔABC is right angled at C, prove that : (a) $\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}$ (b) $\sin(A-B) = \frac{a^2-b^2}{a^2+b^2}$

(iii) If in a ΔABC, two sides are a = 3, b = 5 and $\cos(A - B) = \frac{7}{25}$, find $\tan\frac{C}{2}$.

5. HALF ANGLE FORMULAE :

$s = \frac{a+b+c}{2}$ = semi-perimeter of triangle.

(a) (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$ (iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ (ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{\Delta}{s(s-b)}$ (iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{\Delta}{s(s-c)}$

(d) **Area of Triangle**

$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$, where p_1, p_2, p_3 are altitudes from vertices A,B,C respectively.

Illustration 10 : If in a triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to -

(A) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (B) $\frac{2ab}{a+b}\sin\frac{C}{2}$ (C) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (D) $\frac{b\sin\angle DAC}{\sin(B+C/2)}$

Solution : $\Delta CAB = \Delta CAD + \Delta CDB$

$\Rightarrow \frac{1}{2}ab\sin C = \frac{1}{2}b.CD.\sin\left(\frac{C}{2}\right) + \frac{1}{2}a.CD\sin\left(\frac{C}{2}\right)$

$\Rightarrow CD(a+b)\sin\left(\frac{C}{2}\right) = ab\left(2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\right)$

So $CD = \frac{2ab\cos(C/2)}{(a+b)}$

and in ΔCAD, $\frac{CD}{\sin\angle DAC} = \frac{b}{\sin\angle CDA}$ (by sine rule)

$\Rightarrow CD = \frac{b\sin\angle DAC}{\sin(B+C/2)}$ **Ans. (C,D)**

Illustration 11 : If Δ is the area and 2s the sum of the sides of a triangle, then show $\Delta \le \frac{s^2}{3\sqrt{3}}$.

Solution : We have, $2s = a + b + c$, $\Delta^2 = s(s-a)(s-b)(s-c)$

Now, A.M. $\ge$ G.M.

$$\frac{(s-a)+(s-b)+(s-c)}{3} \ge \{(s-a)(s-b)(s-c)\}^{1/3}$$

or $$\frac{3s-2s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$$

or $$\frac{s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$$

or $$\frac{\Delta^2}{s} \le \frac{s^3}{27} \quad \Rightarrow \quad \Delta \le \frac{s^2}{3\sqrt{3}}$$ **Ans.**

Do yourself - 5 :

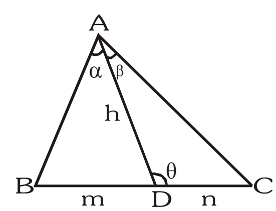
(i) Given a = 6, b = 8, c = 10. Find

(a) sinA (b) tanA (c) $\sin\frac{A}{2}$ (d) $\cos\frac{A}{2}$ (e) $\tan\frac{A}{2}$ (f) Δ

(ii) Prove that in any ΔABC, $(abcs)\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2} = \Delta^2$.

(iii) Show that if $\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right) = \frac{2}{3}\cot\frac{B}{2}$, then a, b, c are in A.P.

6. m-n THEOREM :



$(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

$(m + n)\cot\theta = n\cot B - m\cot C.$

Illustration 12 : The base of a Δ is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that : $\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right) = 4\left(1+\frac{1}{t_2^2}\right)$

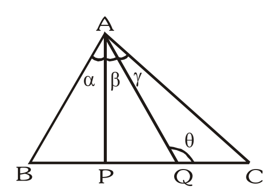
Solution : Let the points P and Q divide the side BC in three equal parts :

Such that BP = PQ = QC = x

Also let,

$\angle BAP = \alpha, \angle PAQ = \beta, \angle QAC = \gamma$

and $\angle AQC = \theta$



From question, $\tan\alpha = t_1$, $\tan\beta = t_2$, $\tan\gamma = t_3$.

Applying

m : n rule in triangle ABC we get,

$(2x + x)\cot\theta = 2x\cot(\alpha + \beta) - x\cot\gamma$ .... (i)

from ΔAPC, we get

$(x + x)\cot\theta = x\cot\beta - x\cot\gamma$ .... (ii) dividing (i) and (ii), we get

$$\frac{3}{2}=\frac{2\cot(\alpha+\beta)-\cot\gamma}{\cot\beta-\cot\gamma}$$

or $\quad 3\cot\beta-\cot\gamma=\dfrac{4(\cot\alpha.\cot\beta-1)}{\cot\beta+\cot\alpha}$

or $\quad 3\cot^2\beta-\cot\beta\cot\gamma+3\cot\alpha.\cot\beta-\cot\alpha.\cot\gamma=4\cot\alpha.\cot\beta-4$

or $\quad 4+4\cot^2\beta=\cot^2\beta+\cot\alpha\,.\,\cot\beta+\cot\beta\,.\,\cot\gamma+\cot\gamma\,.\,\cot\alpha$

or $\quad 4(1+\cot^2\beta)=(\cot\beta+\cot\alpha)(\cot\beta+\cot\gamma)$

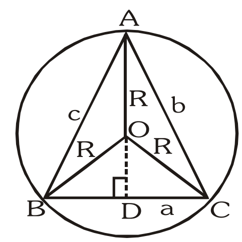
or $\quad 4\left(1+\dfrac{1}{t_2^2}\right)=\left(\dfrac{1}{t_1}+\dfrac{1}{t_2}\right)\left(\dfrac{1}{t_2}+\dfrac{1}{t_3}\right)$

Do yourself - 6 :

(i) The median AD of a ΔABC is perpendicular to AB, prove that $\tan A+2\tan B=0$

7. RADIUS OF THE CIRCUMCIRCLE 'R' :

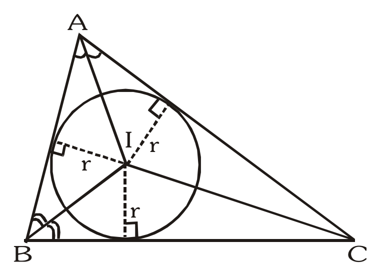
Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.



$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}.$$

8. RADIUS OF THE INCIRCLE 'r' :

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.



$$r=\frac{\Delta}{s}=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$=a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}=b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}}=c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

Illustration 13 : In a triangle ABC, if a : b : c = 4 : 5 : 6, then ratio between its circumradius and inradius is-

(A) $\dfrac{16}{7}$ (B) $\dfrac{16}{9}$ (C) $\dfrac{7}{16}$ (D) $\dfrac{11}{7}$

Solution : $\dfrac{R}{r}=\dfrac{abc}{4\Delta}\Big/\dfrac{\Delta}{s}=\dfrac{(abc)s}{4\Delta^2}\quad\Rightarrow\quad\dfrac{R}{r}=\dfrac{abc}{4(s-a)(s-b)(s-c)}$ ....(i)

$\because\quad a:b:c=4:5:6\quad\Rightarrow\quad\dfrac{a}{4}=\dfrac{b}{5}=\dfrac{c}{6}=k$ (say)

$\Rightarrow\quad a=4k,\ b=5k,\ c=6k$

$\therefore\quad s=\dfrac{a+b+c}{2}=\dfrac{15k}{2},\ s-a=\dfrac{7k}{2},\ s-b=\dfrac{5k}{2},\ s-c=\dfrac{3k}{2}$

using (i) in these values $\dfrac{R}{r}=\dfrac{(4k)(5k)(6k)}{4\left(\frac{7k}{2}\right)\left(\frac{5k}{2}\right)\left(\frac{3k}{2}\right)}=\dfrac{16}{7}$ **Ans. (A)**

Illustration 14 : If A, B, C are the angles of a triangle, prove that : cosA + cosB + cosC = $1+\frac{r}{R}$.

Solution : $\cos A + \cos B + \cos C = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)+\cos C$

$$=2\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2} = 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]$$

$$= 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right] \qquad \left\{\because \frac{C}{2}=90°-\left(\frac{A+B}{2}\right)\right\}$$

$$= 1+2\sin\frac{C}{2}.2\sin\frac{A}{2}.\sin\frac{B}{2} = 1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$$

$$= 1+\frac{r}{R} \qquad \{\text{as, } r = 4R\sin A/2 . \sin B/2 . \sin C/2\}$$

$\Rightarrow$ cosA + cosB + cosC = $1+\frac{r}{R}$. Hence proved.

Do yourself - 7 :

(i) If in ΔABC, a = 3, b = 4 and c = 5, find

(a) Δ (b) R (c) r

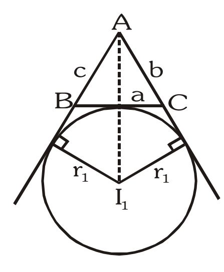
(ii) In a ΔABC, show that :

(a) $\frac{a^2-b^2}{c}=2R\sin(A-B)$ (b) $r\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{\Delta}{4R}$ (c) $a+b+c=\frac{abc}{2Rr}$

(iii) Let Δ & Δ' denote the areas of a Δ and that of its incircle. Prove that Δ : Δ' = $\left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right):\pi$

9. RADII OF THE EX-CIRCLES :

Point of intersection of two external angles and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to ∠A of ΔABC and so on, then -



(a) $r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

(b) $r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$

(c) $r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

I_1, I_2 and I_3 are taken as ex-centre opposite to vertex A, B, C repsectively.

Illustration 15 : Value of the expression $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 0

Solution : $\frac{(b-c)}{r_1}+\frac{(c-a)}{r_2}+\frac{(a-b)}{r_3}$

$\Rightarrow \quad (b-c)\left(\frac{s-a}{\Delta}\right)+(c-a)\left(\frac{s-b}{\Delta}\right)+(a-b)\cdot\left(\frac{s-c}{\Delta}\right)$

$\Rightarrow \quad \frac{(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)}{\Delta}$

$= \frac{s(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta}=\frac{0}{\Delta}=0$

Thus, $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$ **Ans. (D)**

Illustration 16 : If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Solution : We have, $r_1 - r = r_2 + r_3$

$\Rightarrow \quad \frac{\Delta}{s-a}-\frac{\Delta}{s}=\frac{\Delta}{s-b}+\frac{\Delta}{s-c} \quad \Rightarrow \quad \frac{s-s+a}{s(s-a)}=\frac{s-c+s-b}{(s-b)(s-c)}$

$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{2s-(b+c)}{(s-b)(s-c)}$ {as, $2s = a + b + c$}

$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{a}{(s-b)(s-c)} \quad \Rightarrow \quad s^2 - (b + c)s + bc = s^2 - as$

$\Rightarrow \quad s(-a + b + c) = bc \quad \Rightarrow \quad \frac{(b+c-a)(a+b+c)}{2}=bc$

$\Rightarrow \quad (b + c)^2 - (a)^2 = 2bc \quad \Rightarrow \quad b^2 + c^2 + 2bc - a^2 = 2bc$

$\Rightarrow \quad b^2 + c^2 = a^2$

$\therefore \quad \angle A = 90$. **Ans.**

Do yourself - 8 :

(i) In an equilateral ΔABC, $R = 2$, find

(a) r (b) r_1 (c) a

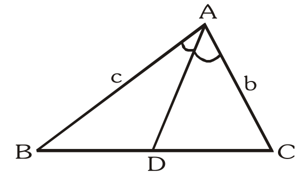
(ii) In a ΔABC, show that

(a) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ (b) $\frac{1}{4}r^2s^2\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=\frac{r+r_1+r_2-r_3}{4\cos C}=R$

(c) $\sqrt{rr_1r_2r_3}=\Delta$

10. ANGLE BISECTORS & MEDIANS :

An angle bisector divides the base in the ratio of corresponding sides.



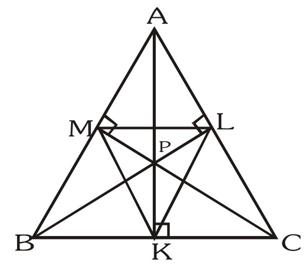
$\frac{BD}{CD}=\frac{c}{b} \quad \Rightarrow \quad BD = \frac{ac}{b+c} \quad \& \quad CD = \frac{ab}{b+c}$

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ and $\beta_a=\frac{2bc\cos\frac{A}{2}}{b+c}$

Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

11. ORTHOCENTRE :



(a) Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.

(b) The distances of the orthocentre from the angular points of the ΔABC are 2R cosA, 2R cosB, & 2R cosC.

(c) The distance of P from sides are 2R cosB cosC, 2R cosC cosA and 2R cosA cosB.

Do yourself - 9 :

(i) If x, y, z are the distance of the vertices of ΔABC respectively from the orthocentre, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$

(ii) If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that

(a) $p_1p_2p_3 = \frac{a^2b^2c^2}{8R^3}$ (b) $\Delta = \sqrt{\frac{1}{2}Rp_1p_2p_3}$

(iii) In a ΔABC, AD is altitude and H is the orthocentre prove that AH : DH = (tanB + tanC) : tanA

(iv) In a ΔABC, the lengths of the bisectors of the angle A, B and C are x, y, z respectively. Show that

$\frac{1}{x}\cos\frac{A}{2} + \frac{1}{y}\cos\frac{B}{2} + \frac{1}{z}\cos\frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Also show that $\frac{a}{b+c} = \sqrt{1 - \frac{x^2}{bc}}$

12. THE DISTANCES BETWEEN THE SPECIAL POINTS :

(a) The distance between circumcentre and orthocentre is $= R\sqrt{1 - 8\cos A\cos B\cos C}$

(b) The distance between circumcentre and incentre is $= \sqrt{R^2 - 2Rr}$

(c) The distance between incentre and orthocentre is $= \sqrt{2r^2 - 4R^2\cos A\cos B\cos C}$

(d) The distances between circumcentre & excentres are

$$OI_1 = R\sqrt{1 + 8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R^2 + 2Rr_1} \text{ \& so on.}$$

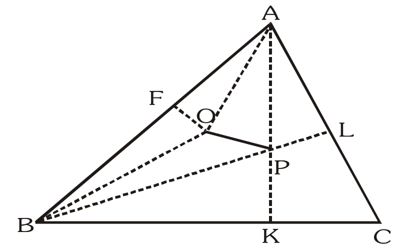
Illustration 17 : Prove that the distance between the circumcentre and the orthocentre of a triangle ABC is $R\sqrt{1 - 8\cos A\cos B\cos C}$.

Solution : Let O and P be the circumcentre and the orthocentre respectively. If OF is the perpendicular to AB, we have ∠OAF = 90 – ∠AOF = 90 – C. Also ∠PAL = 90 – C.

Hence, ∠OAP = A – ∠OAF – ∠PAL = A – 2(90 – C) = A + 2C – 180

$= A + 2C - (A + B + C) = C - B.$

Also OA = R and PA = 2RcosA.



Now in ΔAOP,

$OP^2 = OA^2 + PA^2 - 2OA.\ PA\ \cos OAP$

$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C-B)$

$= R^2 + 4R^2\cos A[\cos A - \cos(C-B)]$

$= R^2 - 4R^2\cos A[\cos(B+C) + \cos(C-B)] = R^2 - 8R^2\cos A\cos B\cos C.$

Hence $OP = R\sqrt{1-8\cos A\cos B\cos C}$. **Ans.**

Do yourself - 10 :

(i) Show that in an equilateral triangle, circumcentre, orthocentre and incentre overlap each other.

(ii) If the incentre and circumcentre of a triangle are equidistant from the side BC, show that $\cos B + \cos C = 1$.

13. SOLUTION OF TRIANGLES :

The three sides a,b,c and the three angles A,B,C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

* If the three sides a,b,c are given, angle A is obtained from $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

or $\cos A = \frac{b^2+c^2-a^2}{2bc}$ .B and C can be obtained in the similar way.

* If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ gives $\frac{B-C}{2}$. Also

$\frac{B+C}{2} = 90° - \frac{A}{2}$, so that B and C can be evaluated. The third side is given by a = b $\frac{\sin A}{\sin B}$

or $a^2 = b^2 + c^2 - 2bc\cos A$.

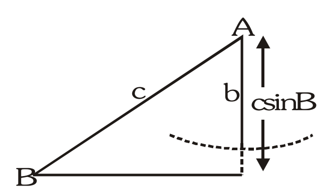
* If two sides b and c and an angle opposite the one of them (say B) are given then

$\sin C = \frac{c}{b}\sin B,\ A = 180° - (B + C)$ and $a = \frac{b\sin A}{\sin B}$ given the remaining elements.

Case I :

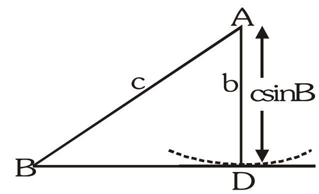
b < c sin B.

We draw the side c and angle B. Now it is obvious from the figure that there is no triangle possible.



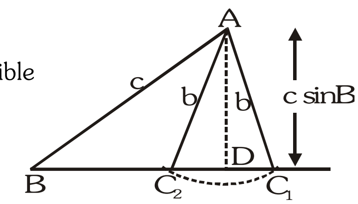
Case II :

$b = c \sin B$ and B is an acute angle, there is only one triangle possible.
and it is right-angled at C.



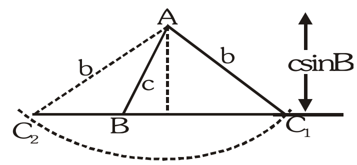
Case III :

$b > c \sin B$, $b < c$ and B is an acute angle, then there are two triangles possible for two values of angle C.



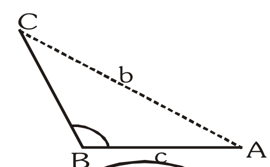
Case IV :

$b > c \sin B$, $c < b$ and B is an acute angle, then there is only one triangle.



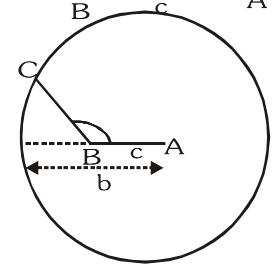
Case V :

$b > c \sin B$, $c > b$ and B is an obtuse angle.
For any choice of point C, b will be greater than c which is a contradication as $c > b$ (given). So there is no triangle possible.



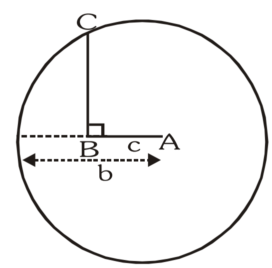
Case VI :

$b > c \sin B$, $c < b$ and B is an obtuse angle.
We can see that the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



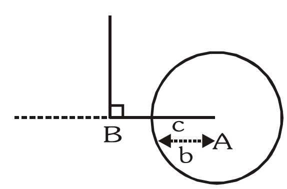
Case VII :

$b > c$ and $B = 90$.
Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



Case VIII :

$b \leq c$ and $B = 90$.
The circle with A as centre and b as radius will not cut the line in any point. So no triangle is possible.



This is, sometimes, called an ambiguous case.

Alternative Method :

By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 - (2c \cos B)a + (c^2 - b^2) = 0 \Rightarrow a = c \cos B \pm \sqrt{(c \cos B)^2 - (c^2 - b^2)}$

$\Rightarrow \quad a = c \cos B \pm \sqrt{b^2 - (c \sin B)^2}$

This equation leads to following cases :

Case-I : If $b < c\sin B$, no such triangle is possible.

Case-II: Let $b = c\sin B$. There are further following case :

(a) B is an obtuse angle $\Rightarrow$ cosB is negative. There exists no such triangle.

(b) B is an acute angle $\Rightarrow$ cosB is positive. There exists only one such triangle.

Case-III: Let $b > c\sin B$. There are further following cases :

(a) B is an acute angle $\Rightarrow$ cosB is positive. In this case triangle will exist if and only if $c\cos B > \sqrt{b^2 - (c\sin B)^2}$ or $c > b \Rightarrow$ Two such triangle is possible. If $c < b$, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cosB is negative. In this case triangle will exist if and only if $\sqrt{b^2 - (c\sin B)^2} > |c\cos B| \Rightarrow b > c$. So in this case only one such triangle is possible. If $b < c$ there exists no such triangle.

This is called an ambiguous case.

* If one side a and angles B and C are given, then A = 180 – (B + C), and $b = \frac{a\sin B}{\sin A}, c = \frac{a\sin C}{\sin A}$.

* If the three angles A,B,C are given, we can only find the ratios of the sides a,b,c by using sine rule (since there are infinite similar triangles possible).

Illustration 18 : In the ambiguous case of the solution of triangles, prove that the circumcircles of the two triangles are of same size.

Solution : Let us say b,c and angle B are given in the ambiguous case. Both the triangles will have b and its opposite angle as B. so $\frac{b}{\sin B} = 2R$ will be given for both the triangles. So their circumradii and therefore their sizes will be same.

Illustration 19 : If a,b and A are given in a triangle and c_1, c_2 are the possible values of the third side, prove that $c_1^2 + c_2^2 - 2c_1c_2\cos 2A = 4a^2\cos^2 A$.

Solution :

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$\Rightarrow c^2 - 2bc\cos A + b^2 - a^2 = 0$.

$c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$.

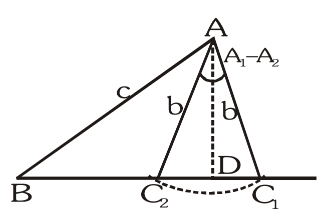
$\Rightarrow c_1^2 + c_2^2 - 2c_1c_2\cos 2A = (c_1 + c_2)^2 - 2c_1c_2(1 + \cos 2A)$

$= 4b^2\cos^2 A - 2(b^2 - a^2)2\cos^2 A = 4a^2\cos^2 A$.

Illustration 20 : If b,c,B are given and $b < c$, prove that $\cos\left(\frac{A_1 - A_2}{2}\right) = \frac{c\sin B}{b}$.

Solution : $\angle C_2AC_1$ is bisected by AD.

$\Rightarrow$ In ΔAC_2D, $\cos\left(\frac{A_1 - A_2}{2}\right) = \frac{AD}{AC_2} = \frac{c\sin B}{b}$



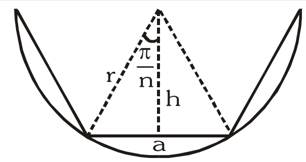
Hence proved.

Do yourself - 11 :

(i) If b,c,B are given and b<c, prove that $\sin\left(\frac{A_1 - A_2}{2}\right) = \frac{a_1 - a_2}{2b}$

(ii) In a ΔABC, b,c,B (c > b) are gives. If the third side has two values a_1 and a_2 such that $a_1 = 3a_2$, show that $\sin B = \sqrt{\frac{4b^2 - c^2}{3c^2}}$.

14. REGULAR POLYGON :



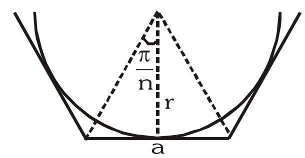
A regular polygon has all its sides equal. It may be inscribed or circumscribed.

(a) Inscribed in circle of radius r :

(i) $a = 2h\tan\frac{\pi}{n} = 2r\sin\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr\sin\frac{\pi}{n}$ and $A = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$

(b) Circumscribed about a circle of radius r :



(i) $a = 2r\tan\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides circumscribed about a given circle of radius r is given by $P = 2nr\tan\frac{\pi}{n}$ and $A = nr^2\tan\frac{\pi}{n}$

Do yourself - 12 :

(i) If the perimeter of a circle and a regular polygon of n sides are equal, then prove that $\frac{\text{area of the circle}}{\text{area of polygon}} = \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$.

(ii) The ratio of the area of n-sided regular polygon, circumscribed about a circle, to the area of the regular polygon of equal number of sides inscribed in the circle is 4 : 3. Find the value of n.

15. IMPORTANT POINTS :

(a) (i) If a cos B = b cos A, then the triangle is isosceles.

(ii) If a cos A = b cos B, then the triangle is isosceles or right angled.

(b) In right angle triangle

(i) $a^2 + b^2 + c^2 = 8R^2$ (ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$

(c) In equilateral triangle

(i) $R = 2r$ (ii) $r_1 = r_2 = r_3 = \frac{3R}{2}$

(iii) $r : R : r_1 = 1 : 2 : 3$ (iv) area $= \frac{\sqrt{3}a^2}{4}$ (v) $R = \frac{a}{\sqrt{3}}$

(d) (i) The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.

(ii) The orthocentre of right angled triangle is the vertex at the right angle.

(iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1 except in case of equilateral triangle. In equilateral triangle, all these centres coincide

(e) Area of a cyclic quadrilateral $= \sqrt{s(s-a)(s-b)(s-c)(s-d)}$

where a, b, c, d are lengths of the sides of quadrilateral and $s = \frac{a+b+c+d}{2}$.

Illustration 21 : For a ΔABC, it is given that $\cos A + \cos B + \cos C = 3/2$. Prove that the triangle is equilateral.

Solution : If a, b, c are the sides of the ΔABC, then $\cos A + \cos B + \cos C = 3/2$

$$\Rightarrow \quad \frac{b^2+c^2-a^2}{2bc}+\frac{a^2+c^2-b^2}{2ac}+\frac{a^2+b^2-c^2}{2ab}=\frac{3}{2}$$

$$\Rightarrow \quad ab^2 + ac^2 - a^3 + bc^2 + ba^2 - b^3 + ca^2 + cb^2 - c^3 = 3abc$$

$$\Rightarrow \quad ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - 6abc = a^3 + b^3 + c^3 - 3abc$$

$$\Rightarrow \quad a(b-c)^2+b(c-a)^2+c(a-b)^2=\frac{(a+b+c)}{2}\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}$$

$$\Rightarrow \quad (a+b-c)(a-b)^2 + (b+c-a)(b-c)^2 + (c+a-b)(c-a)^2 = 0 \quad \ldots\ldots\ldots \text{(i)}$$

as we know $a + b > c$, $b + c > a$, $c + a > b$

$\therefore$ each term on the left side of equation (i) has positive coefficient multiplied by perfect square, each must be separately zero.

$\Rightarrow$ $a = b = c$.

Hence Δ is equilateral. **Ans.**

Illustration 22 : In a triangle ABC, if $\cos A + 2\cos B + \cos C = 2$. Prove that the sides of the triangle are in A.P.

Solution : $\cos A + 2\cos B + \cos C = 2$ or $\cos A + \cos C = 2(1 - \cos B)$

$$\Rightarrow \quad 2\cos\left(\frac{A+C}{2}\right).\cos\left(\frac{A-C}{2}\right)=4\sin^2 B/2$$

$$\Rightarrow \quad \cos\left(\frac{A-C}{2}\right)=2\sin\frac{B}{2} \qquad \left\{\text{as } \cos\left(\frac{A+C}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)=\sin\frac{B}{2}\right\}$$

$$\Rightarrow \quad \cos\left(\frac{A-C}{2}\right)=2\cos\left(\frac{A+C}{2}\right)$$

$$\Rightarrow \quad \cos\frac{A}{2}.\cos\frac{C}{2}+\sin\frac{A}{2}.\sin\frac{C}{2}=2\cos\frac{A}{2}.\cos\frac{C}{2}-2\sin\frac{A}{2}.\sin\frac{C}{2}$$

$$\Rightarrow \quad \cot\frac{A}{2}.\cot\frac{C}{2}=3 \quad \Rightarrow \quad \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=3$$

$$\Rightarrow \quad \frac{s}{(s-b)}=3 \Rightarrow s=3s-3b \Rightarrow \quad 2s = 3b$$

$\Rightarrow$ $a + c = 2b$, $\quad\therefore$ a, b, c are in A.P. **Ans.**

ANSWERS FOR DO YOURSELF

1 : (i) 90

4 : (iii) $\frac{1}{3}$

5 : (i) (a) $\frac{3}{5}$ (b) $\frac{3}{4}$ (c) $\frac{1}{\sqrt{10}}$ (d) $\frac{3}{\sqrt{10}}$ (e) $\frac{1}{3}$ (f) 24

7 : (i) (a) 6 (b) $\frac{5}{2}$ (c) 1

8 : (i) (a) 1 (b) 3 (c) $2\sqrt{3}$

12 : (ii) 6